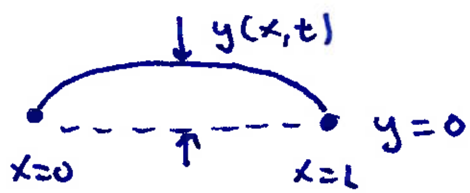
Wave Equation

models a vibrating string

string of length L, held tight, ends fixed

natural position



$y(x,t)$: displacement from natural position

the string naturally "wants" to be flat and will provide a restoring acceleration / force

if displaced like this :  $y_{xx} < 0$, string "fights" back w/ a downward acceleration $\rightarrow y_{tt} < 0$

if displaced like this:  $y_{xx} > 0$, ~~String~~ string has an <u>upward</u> acceleration $\rightarrow$ $y_{tt} > 0$

acceleration and concavity have the sign.

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}}$$

<u>1-D Wave</u> Eq.

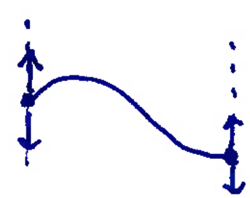
or $\boxed{y_{tt} = a^2 y_{xx}}$

a^2 determines how fast string vibrates (wave speed)

for string: $a^2 = \dfrac{\text{tension}}{\text{density}}$

Boundary conditions: $y(0,t) = y(L,t) = 0$ ends fixed at 0 (Dirichlet)

or $y_x(0,t) = y_x(L,t) = 0$ ends can move vertically but not horizontally (Neumann)



Initial Conditions: because 2nd-order deriv. with time

$y(x,0) = f(x)$ initial displacement (plucking)

$y_t(x,0) = g(x)$ initial velocity (strumming)

today we will solve the problem w/ ends fixed at 0 and no initial velocity ("Problem A")

$y_{tt} = a^2 y_{xx}$ $\quad 0 < x < L \quad t > 0$

$y(0,t) = y(L,t) = 0$ ends fixed at 0

$y_t(x,0) = g(x) = 0$ no initial velocity

$y(x,0) = f(x)$ initial displacement

we will again use the method of separation of variables

$y(x,t) = X(x)T(t)$

$y_{xx} = X''T \qquad y_{tt} = XT''$

wave eq: $X T'' = a^2 X'' T$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = \text{separation constant} = -\lambda \qquad \text{(just like w/ heat eq.)}$$

we get two ODEs:

$$X'' + \lambda X = 0$$

$$T'' + a^2 \lambda T = 0$$

BC: $y(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$

$y(L,t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$

X solution is the same as non-insulated heat eq.

⋮

$$\boxed{\lambda_n = \frac{n^2 \pi^2}{L^2}} \qquad n = 1, 2, 3, \ldots \qquad \boxed{X_n = \sin\left(\frac{n\pi x}{L}\right)}$$

eigenvalues

eigenfunctions

$$T'' + a^2 \lambda T = 0$$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

$$T(t) = A \cos\left(\frac{n\pi a}{L} t\right) + B \sin\left(\frac{n\pi a}{L} t\right)$$

IC: $y_t(x, 0) = 0$ no initial velocity

$$X(x) T'(0) = 0 \rightarrow T'(0) = 0$$

$\vdots$

$B = 0$

$$\boxed{T_n = \cos\left(\frac{n\pi a}{L} t\right)}$$ $n = 1, 2, 3, \ldots$

for wave, the time solution is also periodic

for each n, there is one solution $y_n = X_n T_n$

each of these is called a "mode" or "harmonic" general solution:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)}$$

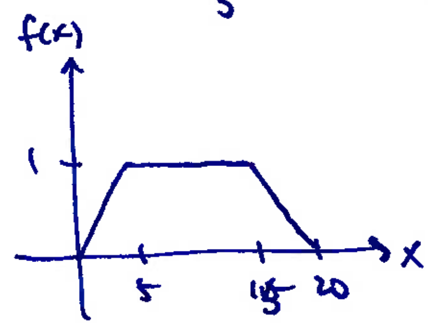
last IC: $y(x,0) = f(x)$ initial displacement

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{sine series}$$

$$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

<u>example</u> $L = 20$, $a = 1$, $g(x) = 0$ no initial velocity

initial displacement: $f(x) = \begin{cases} \frac{1}{5} x & 0 < x < 5 \\ 1 & 5 < x < 15 \\ \frac{20-x}{5} & 15 < x < 20 \end{cases}$



$$y(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)\right]\cos\left(\frac{n\pi}{20}t\right)\sin\left(\frac{n\pi}{20}x\right)$$

$n=1$ Fundamental mode / harmonic

$$y_1(x,t) = \frac{8\sqrt{2}}{\pi^2}\cos\left(\boxed{\frac{\pi}{20}}t\right)\sin\left(\frac{\pi}{20}x\right)$$

↳ frequency of vibration (sound)

$\frac{\pi}{20}$ rad/s $\Rightarrow \frac{\pi/20}{2\pi} = \frac{1}{40}$ Hz

$n=2$ second harmonic

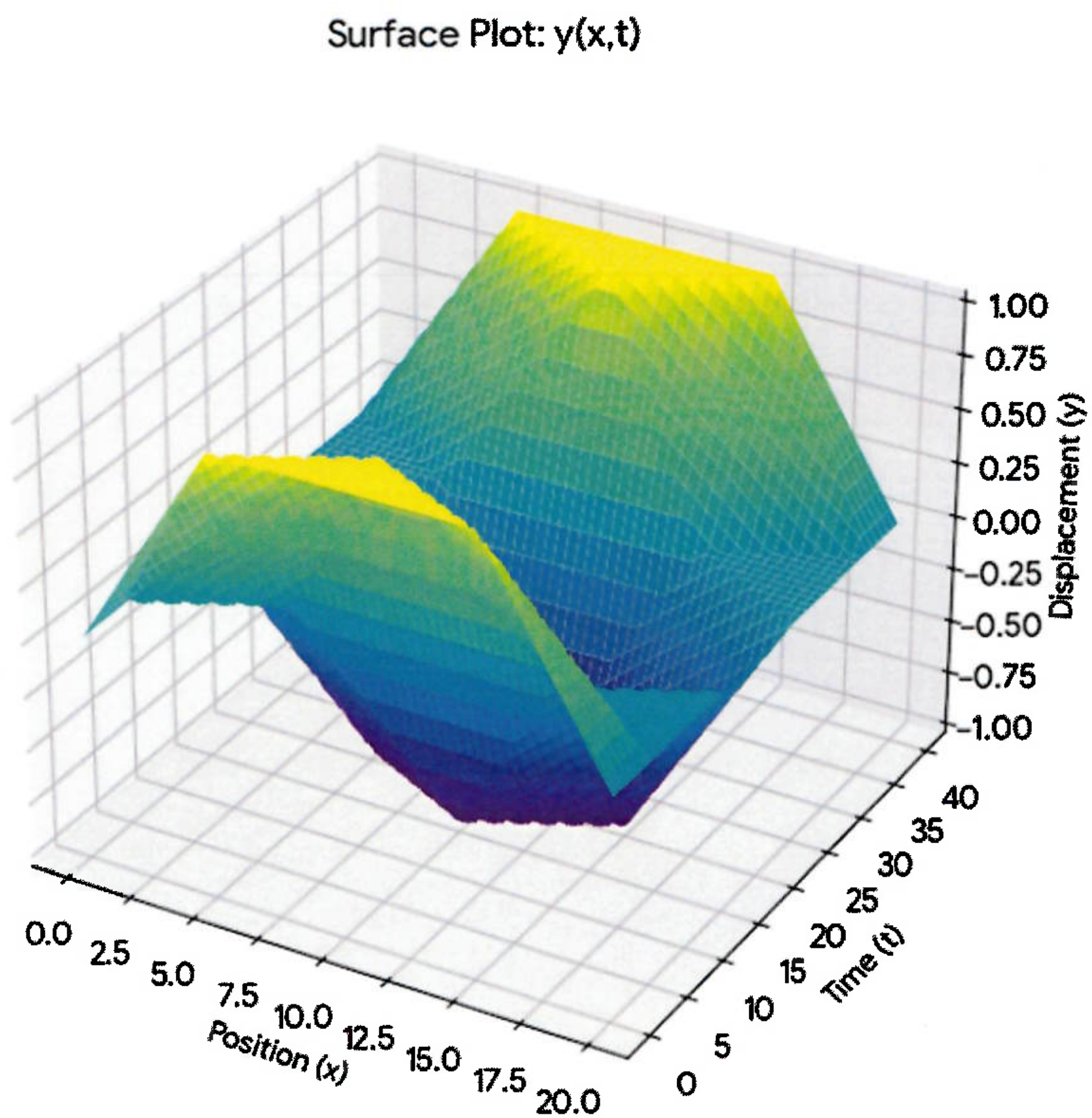
freq: $2 \cdot \frac{\pi}{20} = \frac{1}{20}$ Hz (doubles first harmonic)

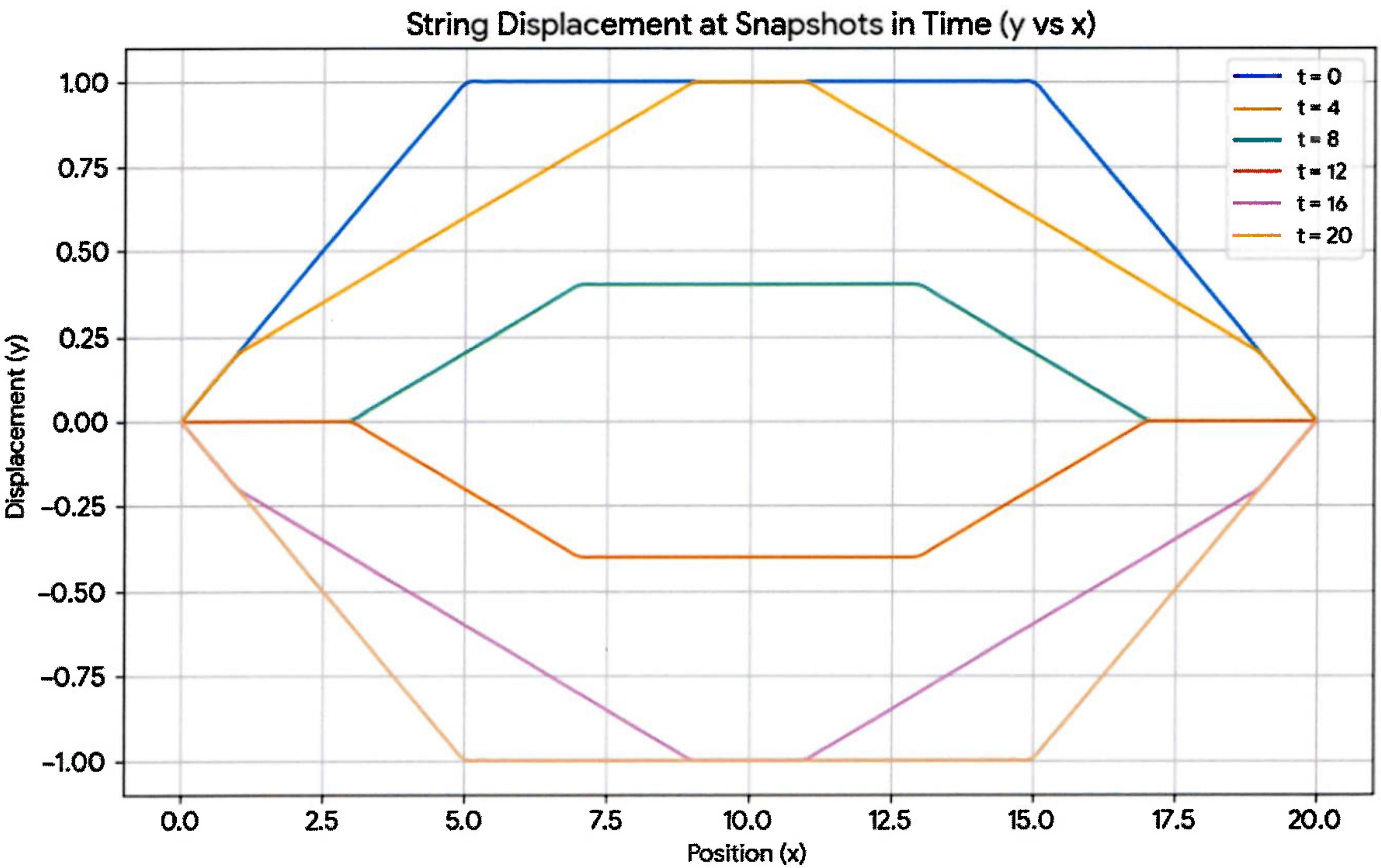
↳ an octave higher

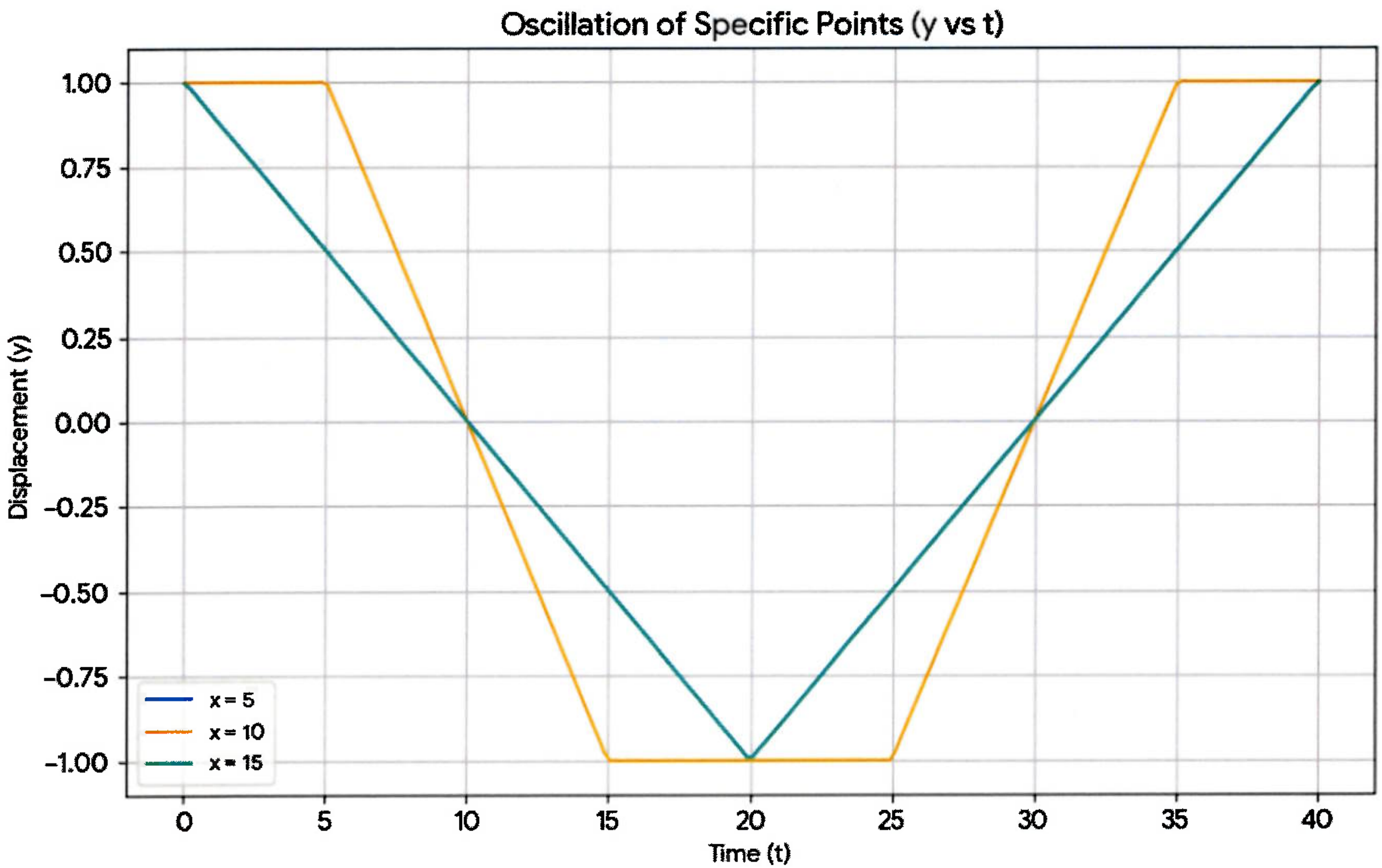
$n=3$ 3rd harmonic

freq: $3 \cdot \frac{\pi}{20} = \frac{3}{40}$ Hz (one octave and a perfect fifth above fundamental)

what we hear is combination of ALL n's


Surface Plot: y(x,t)
Position (x)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Time (t)
0
5
10
15
20
25
30
35
40
Displacement (y)
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00


String Displacement at Snapshots in Time (y vs x)
Displacement (y)
Position (x)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t = 0
t = 4
t = 8
t = 12
t = 16
t = 20



Oscillation of Specific Points (y vs t)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Displacement (y)
0
5
10
15
20
25
30
35
40
Time (t)
x = 5
x = 10
x = 15